

A PROBLEM ON THE TRIANGLE

by

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A thesis submitted to the Department of
Mathematics and the Faculty of
the Graduate School in partial
fulfillment of the require-
ments for the Master's
degree.

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A PROBLEM ON THE TRIANGLE.

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Introduction.

In this paper it is proposed to investigate the triangles formed by joining three points of intersection of the interior and the exterior trisectors of a given finite triangle.

The following problem was proposed by Prof. Archibald of Brown University: "Let ABC be a triangle with AP and AR trisectors of angle BAC ; BP and BX trisectors of angle ABC ; CX and CR trisectors of angle ACB . Draw PR , RX , and PX ; to prove that PRX is an equilateral triangle".

A geometrical solution of this problem was attempted by A.H.Holmes and published in the American Mathematical Monthly, Dec.1910, and a trigonometrical solution by E.M. Morgan was published in the same magazine June 1914.

It is our purpose to discuss the above problem and its solutions and to generalize them. Besides internal trisectors, external trisectors may be considered. By combining their points of intersection in various ways, a very large number of triangles may be obtained. Only one equilateral triangle has been found besides the one given in the problem by Archibald. Various methods of elimination are introduced to reduce the number of triangles

which can be equilateral.

We will not consider triangles which have two vertices on the same trisector; nor triangles which have one vertex coincident with a vertex of the original triangle ABC.

Every point of intersection of two trisectors belongs to one of the three sides of triangle ABC; that is, every point lies on only two trisectors which come from the extremities of one of the three sides.

The points of intersection will be classified in this manner:

1. Intersection of interior trisectors. No. of points.

a_1 . Both adjacent to their side. 1

b_1 . One adjacent and one non-adjacent. 2

(By a non-adjacent trisector is meant that an adjacent trisector lies between it and its side: e.g. Trisector (1), see Fig.2, is adjacent to AB and (2) is non-adjacent to AB. This is reversed for side AC.)

c_1 . Both non-adjacent. 1

2. Intersection of exterior trisectors.

a_2 . Both adjacent. 1

b_2 . One adjacent and one non-adjacent. 2

c_2 . Both non-adjacent. 1

3. Intersection of interior with exterior trisectors.

a_3 .Both adjacent	2
b_3 .Interior non-adjacent and exterior adjacent	2
c_3 .Interior adjacent and exterior non-adjacent	2
d_3 .Both non-adjacent.	<u>2</u>
Total number of points for a side	16

As there are sixteen points for each side there are forty eight points in all.

Notation: We shall use a, b, c ; etc., to designate types of intersections of interior trisectors, - both adjacent to their sides, one adjacent and one non-adjacent, both non-adjacent.

All triangles having for vertices, two or three points of intersection which belong to the same side of the triangle ABC, will not be considered.

In this paper we have investigated only the case where the given triangle ABC is finite.

The triangle ABC having an infinite vertex offers an interesting field of work.

Table of Alignment.

It has been found very convenient to have a table indicating at a glance which vertices are on the same trisector and which are not.

This is what we have called a table of alignment and is given below.

The following notation is used for figures 1,2,3. Each point of intersection lies on the two trisectors indicated by the first row and the first column.

	1	2	3	4	5	6	1'	2'	3'	4'	5'	6'
1	A	A	M	N	O	P	A	A	M ₁	N ₁	O ₁	P ₁
2	A	A	R	S	T	J	A	A	R ₁	S ₁	T ₁	J ₁
3	M	R	C	C	K	L	M ₂	R ₂	C	C	K ₁	L ₁
4	N	S	C	C	X	Z	N ₂	S ₂	C	C	X ₁	Z ₁
5	O	T	K	X	B	B	O ₂	T ₂	K ₂	X ₂	B	B
6	P	J	L	Z	B	B	P ₂	J ₂	L ₂	Z ₂	B	B
1'	A	A	M ₁	N ₂	O ₂	P ₁	A	A	M'	N'	O'	P'
2'	A	A	R ₂	S ₂	T ₂	J ₂	A	A	R'	S'	T'	J'
3'	M ₁	R ₁	C	C	K ₂	L ₂	M'	R'	C	C	K'	L'
4'	N ₁	S ₁	C	C	X ₂	Z ₂	N'	S'	C	C	X'	Z'
5'	O ₁	T ₁	K ₁	X ₁	B	B	O'	T'	K'	X'	B	B
6'	P ₁	J ₁	L ₁	Z ₁	B	B	P'	J'	L'	Z'	B	B

Classification of the Points of Intersection
in Columns by Types and in Rows by the Sides to
Which They Belong.

This table is to aid in elimination. All triangles having two or three vertices on the same trisector are to be omitted.

By the preliminary theorems of elimination, certain points may become infinite. By the theorems of elimination we may eliminate certain types of intersections as vertices of our triangles.

By this table we see, for example, that X is the intersection of the trisectors 4 and 5 which are both interior trisectors adjacent to their side BC.

Interior			Exterior		
2adj.	1adj. 1non-adj.	2non-adj.	2adj.	1adj. 1non-adj.	2non-adj.
a_1	b_1	c_1	a_1	b_1	c_1
BC X 45	K 35 Z 46	L 36	X' 4'5'	K' 3'5' Z' 4'6'	L' 3'6'
AC R 23	M 13 S 24	N 14	R' 2'3'	M' 1'3' S' 2'4'	N' 1'4'
AB P 16	J 26 O 15	T 25	P' 1'6'	J' 2'6' O' 1'5'	T' 2'5'

Exterior and Interior.

	2 adj.	1 exterior adj.	1 ext. non-adj.	2 non-adj.
	a_3	b_3	c_3	d_3
BC	X_{45^1}	K_{35^1}	$K_{3^5 2}$	L_{36^1}
	$X_{4^5 2}$	$Z_{4^1 6^2}$	$Z_{4^6 1}$	$L_{3^1 6^2}$
AC	$R_{2^3 1}$	$M_{1^3 1}$	$M_{1^3 2}$	$N_{1^4 1}$
	$R_{2^1 3}$	$S_{2^1 4}$	$S_{2^4 1}$	$N_{1^4 2}$
AB	P_{16^1}	J_{26^1}	$J_{2^1 6^2}$	T_{25^1}
	$P_{1^6 2}$	$O_{1^5 2}$	$O_{1^5 1}$	$T_{2^1 5^2}$

Theorems Preliminary to the Theorems of Elimination.

Theorem 1.

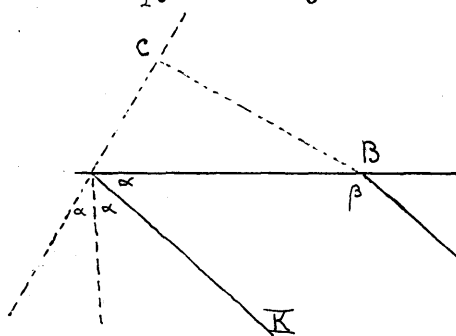
No intersection of interior trisectors can lie at infinity.

Obviously all these points lie within the original finite triangle ABC.

Theorem 2.

Of the intersections of exterior trisectors, only the intersection of the two non-adjacent trisectors can lie at infinity. There are three possible types to consider, namely a_2 , b_2 , and c_2 .

a_2 . Two adjacent exterior trisectors must intersect.



Proof: Let AB be a side of triangle ABC, and AK and BL the two exterior adjacent trisectors.

If AK and BL are parallel,

$$\angle \alpha + \beta = 180^\circ.$$

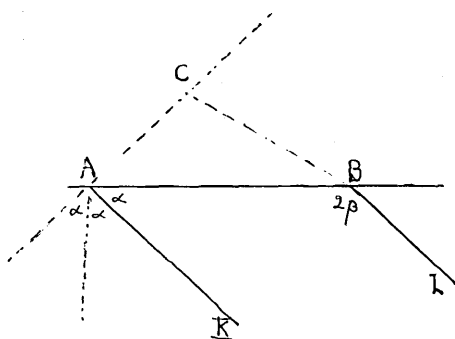
$$\text{Then } 3(\alpha + \beta) = 540^\circ.$$

$$\text{But } 3\alpha + 3\beta + A + B = 360^\circ.$$

$\therefore 3(\alpha + \beta) < 540^\circ$, and AK cannot be parallel to BL.

b_2 . An adjacent and a non-adjacent exterior trisector must intersect.

Proof: Let AB be a side of triangle ABC; and the



parallel lines AK and BL be the adjacent and the non-adjacent exterior trisectors respectively. 2β must be the obtuse angle, for otherwise the other trisector at B

would fall on the other side of AB and cut angle BAK: i.e. AK would not be the trisector adjacent to AB.

If AK is parallel to BL,

$$\alpha + 2\beta = 180^\circ$$

$$3\beta + B = 180^\circ$$

$$3\alpha + A = 180^\circ$$

$$3\alpha + 6\beta = 540^\circ$$

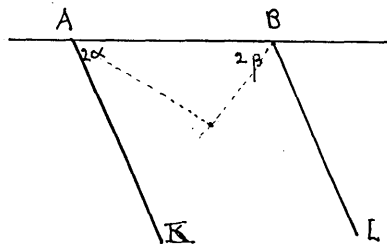
$$3\alpha + 3\beta = 540^\circ - 180^\circ + B = 360^\circ + B, \quad B \neq 0.$$

$$\text{But as } 3(\alpha + \beta) + A + B = 360^\circ,$$

$$3(\alpha + \beta) > 360^\circ \text{ is impossible.}$$

$$\alpha + 2\beta \neq 180^\circ, \text{ and AK and BL must intersect.}$$

c₂. Two non-adjacent trisectors may be parallel.



Proof: Let AB be a side of triangle ABC, and AK and BL be the two non-adjacent trisectors. If AK is parallel to BL,

$$2\alpha + 2\beta = 180^\circ,$$

$$3(\alpha + \beta) = 270^\circ,$$

$$3(\alpha + \beta) + A + B = 360^\circ \quad \text{if } \alpha \text{ and } \beta < 60^\circ.$$

Therefore it is possible to construct a triangle with one side having two non-adjacent trisectors parallel.

$$A + B = 360^\circ - 270^\circ = 90^\circ. \quad \therefore \text{angle } C = 90^\circ.$$

It is impossible, therefore, to have the non-adjacent trisectors of more than one side parallel if triangle ABC is finite.

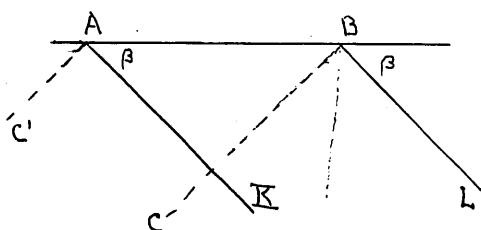
Therefore only the intersection of two non-adjacent trisectors of the intersections of exterior trisectors can lie at infinity.

Theorem 3.

Of the intersections of interior trisectors with exterior trisectors, only the intersections of type b may, in a special case, become infinite.

There are the four possible types to consider, a_3 , b_3 , c_3 , d_3 .

a_3 . An adjacent exterior and an adjacent interior trisector must intersect if ABC is finite.
may be parallel if one vertex is infinite.



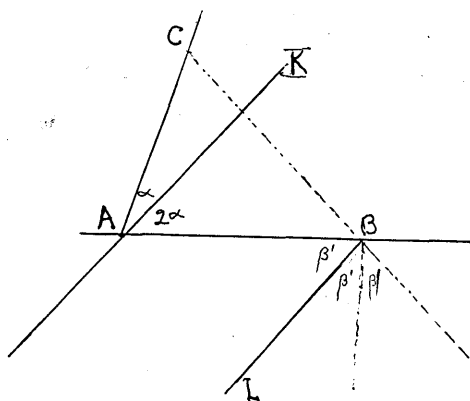
Proof: Let AB be a side of triangle ABC, and let AK and BL be the adjacent interior and exterior trisectors respectively.

Draw CB making an angle of 2β with BL. As AK and BL are parallel, angle KAB equals β .

Draw AC' making an angle of 2β with AK.
AC is then parallel to BL.

The third vertex C lies at infinity and we see that no finite triangle may have an adjacent exterior and an adjacent interior trisector parallel.

b_3 . A non-adjacent interior and an adjacent exterior trisector may be parallel.



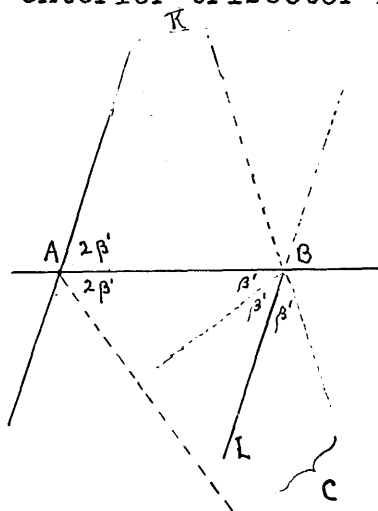
Proof: Let AB be a side of triangle ABC, having two parallel lines AK and BL as the non-adjacent interior and the adjacent exterior trisectors respectively.

AC will make an angle α with AK, and BC an angle $2\beta'$ with BL.

$$\text{The angle } C = 3\alpha = 3/2\beta'$$

Therefore the non-adjacent interior and the adjacent exterior trisectors, of an equal side of an isosceles triangle are parallel.

c_3 . An adjacent interior and a non-adjacent exterior trisector must intersect.



Proof; Let AB be a side of triangle ABC, having the two parallel lines AK and BL as its adjacent interior and non-adjacent exterior trisectors, respectively.

The side BC must make an angle $= 2\beta'$ with BL.

As AK is the adjacent interior trisector, it will make an angle of $4\beta'$ with the side AC.

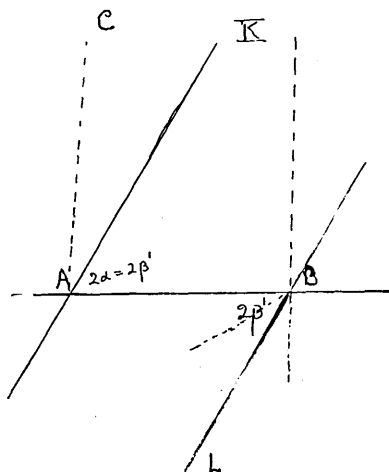
But if AC makes an angle with AK greater than β' it must cut BC on the side of AB which will make AK an exterior trisector and BL interior, contrary to the hypothesis.

Therefore an adjacent interior and a non-adjacent exterior trisector cannot be parallel.

d_3 . An interior non-adjacent and an exterior non-adjacent trisector must intersect if triangle may be parallel if a vertex

ABC is finite.
of triangle ABC is infinite.

Proof: Let AB be a side of triangle ABC, having parallel lines AK and BL as the interior non-adjacent and



the exterior non-adjacent trisectors respectively.

BC will make an angle β' with BL, and AC an angle with AK.

But $\alpha = \beta'$

As AC and BC are parallel, C lies at infinity. Only the triangle with one infinite

vertex has a point of type d_3 at infinity.

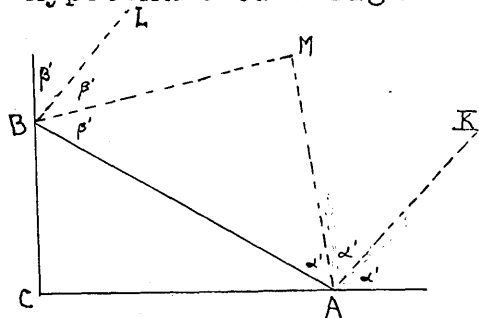
Therefore if triangle ABC is finite, all points of intersection of type d are finite.

Therefore of the intersections of interior trisectors with exterior trisectors, only the intersections of type b_3 may, in a special case, become infinite.

Q.E.D.

Theorem 4.

The two exterior trisectors non-adjacent to the hypotenuse of a right triangle are parallel.



Given any right triangle ABC,

To prove that the two exterior trisectors BL and AK non-adjacent to the hypotenuse AB, are parallel.

Proof: Any exterior angle equals the sum of the two opposite interior angles.

$$\therefore 3\alpha' + 3\beta' = 2C + \angle A + \angle B = 3 \text{ right angles.}$$

$$\therefore 2(\alpha' + \beta') = 180^\circ \text{ and BL is parallel to AK.}$$

Q.E.D.

Theorem 5.

A triangle may be so constructed as to have a maximum of

1. Six infinite points of intersection of type b_3 .
- or 2. One infinite point of type c_2 , and
two infinite points of type b_3 .

1. A triangle may be so constructed as to have the maximum number, six, of infinite points of type b_3 .

Proof: By Theorem 3, the intersection of an interior trisector non-adjacent to, and an exterior trisector adjacent to an equal side of an isosceles triangle, lies at infinity.

From the symmetry of the isosceles triangle, there will be two infinite points of this kind.

As the equilateral triangle may be regarded as triply isosceles, there will be six infinite points of type b_3 .

2. A triangle may be constructed so as to have the maximum number, three, of infinite points, two of type b_3 and one of type c_2 .

Proof:

By combining the requirements for infinite points given in Theorems 2 and 3, we have the isosceles right triangle.

The triangle ABC, Figure 2, has $\angle A = \angle C = 45^\circ$, and $\angle B = 90^\circ$.

There are two infinite points of type b_3 , (as triangle ABC is isosceles; see Preliminary Theorem 3, b_3); and there is one infinite point of type c_2 , (as triangle ABC is right; see Preliminary Theorem 2, c_2 .)

Theorems of Elimination.

Theorem 1.

Types of triangles which, in special cases, may have one vertex at infinity cannot be equilateral.

Proof: If two vertices are finite and the third lies at infinity we should have a triangle with one side finite in length and two sides infinite in length.

Therefore the triangle cannot be equilateral.

Theorem 2.

Types of triangles which have as two vertices, points of type c_1 or b_3 cannot be equilateral.

(Triangle ABC must be finite)

Proof: From Preliminary Theorems 2 and 3, we see that there are only two types of points of intersection which can be made to lie at infinity: i.e. c_1 and b_3 .

If we can show that one triangle of each type is non-equilateral, we can conclude that the type is non-equilateral.

We must consider all possible types having two vertices which, in a special case, may become infinite. We have the following possible types to consider:

1. Type $c_2 c_2 x$,
2. Type $b_3 b_3 x$,
3. Type $c_2 b_3 x$, where x represents an arbitrary point.

1. Type $c_2 c_2 x$ is not equilateral.

In order for c_2 to become infinite triangle ABC must be right by Preliminary Theorem 2, c_2 .

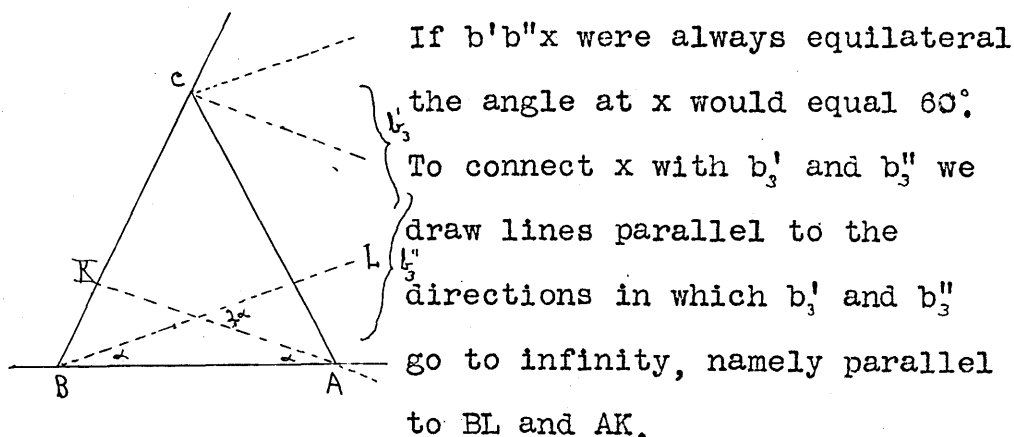
Construct an arbitrary right triangle. All points but one of type c_2 will be finite.

Therefore a triangle of type $c_2 c_2 x$ is non-equilateral.

2. Type $b_3 b_3 x$ is non-equilateral.

In order for b_3 to become infinite triangle ABC must be isosceles, by Preliminary Theorem 3, b_3 .

Construct an arbitrary isosceles triangle ABC. All but two points of type b_3 will be finite.



Then $\angle x = 2\alpha$ or $180^\circ - 2\alpha$.

as $\angle x \neq 60^\circ$, triangle $b_3'b_3''x$ is not equilateral.

3. Type $c_2 b_3 x$ is non-equilateral.

Construct an arbitrary right triangle.

The c_1 alone will be infinite.

Triangles $c_2 b_3 x$ are non-equilateral.

Therefore types of triangles which have as two vertices points of type c_2 or b_3 cannot be equilateral.

Q.E.D.

Theorem 3.

Types of triangles having as vertices points of type c_2 or b_3 , are non-equilateral.

The following include all such possible types,-

1. Type $c_2 c_2 c_2$,

2. Type $b_3 b_3 b_3$,

3. Type $c_2 c_2 b_3$,

4. Type $c_2 b_3 b_3$,

1. Type $c_2 c_2 c_2$ is non-equilateral.

In order that a point of type c_2 be infinite triangle ABC must be right.

A finite triangle can have only one right angle.

Construct an arbitrary right triangle.

Only one point, of type c_2 , is infinite.

Therefore $c_2 c_2 c_2$ is non-equilateral.

2. Type $b_3 b_3 b_3$ is non-equilateral.

In order that a point of type b_3 be infinite triangle ABC must be isosceles.

Construct an arbitrary isosceles triangle. Only two points, of type b_3 , are infinite.

By Theorem 2, type $b_3 b_3 x$ is non-equilateral.

Therefore type $b_3 b_3 b_3$ is non-equilateral, as it is a special case of type $b_3 b_3 x$.

3. Type $c_2 c_2 b_3$ is non-equilateral.

This is a special case of type $c_2 b_3 x$, or of type $c_2 c_2 x$.

4. Type $c_2 b_3 b_3$ is non-equilateral.

This may be regarded as a special case of type $c_2 b_3 x$, or of type $b_3 b_3 x$.

Therefore types of triangles having as vertices, points of type c_2 or b_3 , are non-equilateral.

Q.E.D.

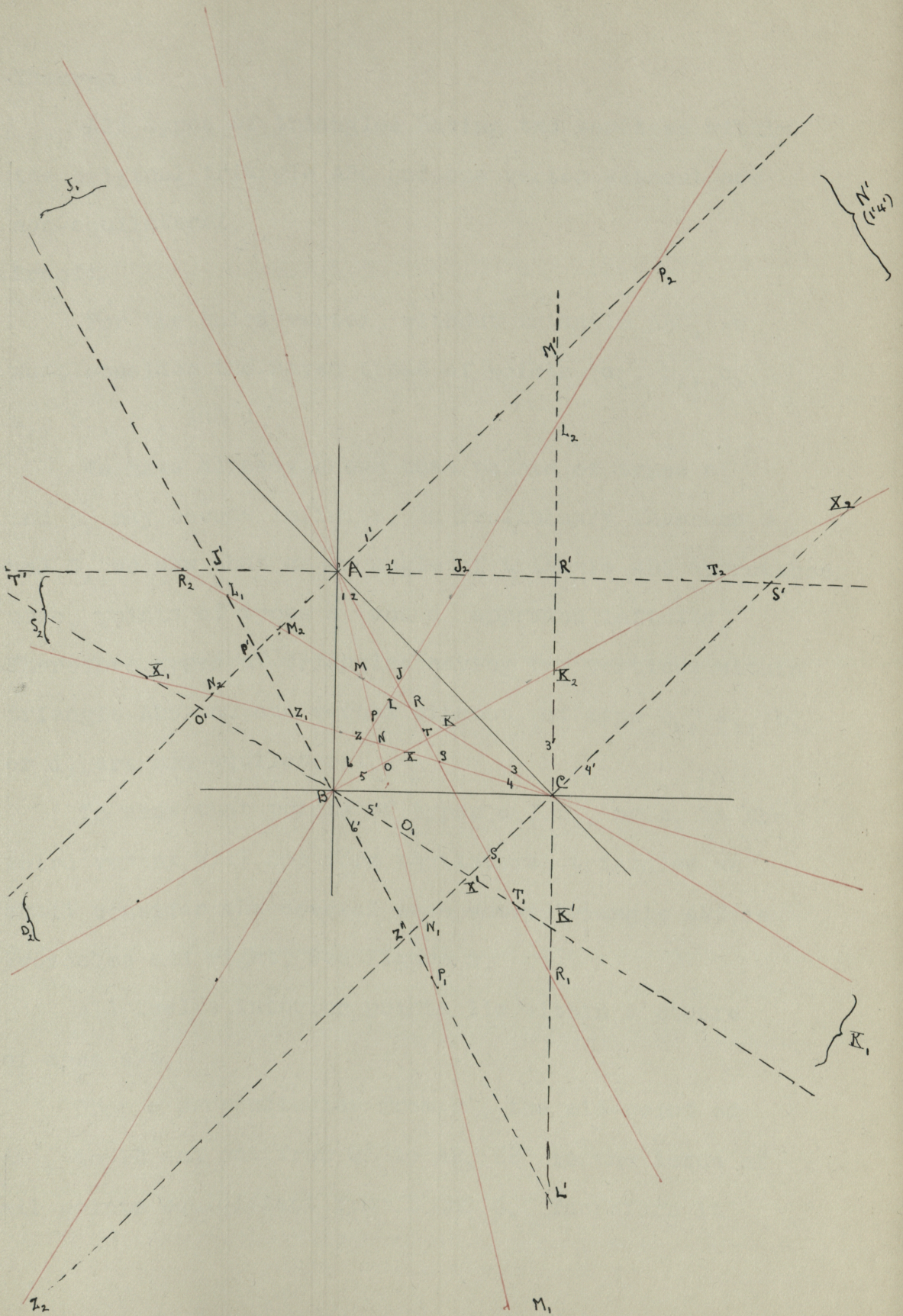


Figure II.

Theorem 4.

All types of triangles having two vertices within the original triangle ABC and one vertex without are non-equilateral.

Proof:

For the third vertex, without triangle ABC, we must consider the seven types of points, a_2 , b_2 , c_2 , a_3 , b_3 , c_3 , and d_3 .

We have already shown that points of types c_2 and b_3 may become infinite. In Preliminary Theorems 2 and 3, we saw that as an angle of triangle ABC approaches zero, points of types a and d approach infinity. Therefore types of triangles having two vertices within triangle ABC and one vertex without, of type c_2 , a_3 , b_3 , or d_3 are non-equilateral.

To show that points of types a_2 , b_2 , or c_3 as the third vertex will not form equilateral triangles we shall consider the special case where triangle ABC is isosceles and right. See Figure 2.

All twelve interior points lie within a square of area $a^2/4$.

Drop a perpendicular from E' , the mid-point of AR' , to BC and one from Q' to AB. EE' is the locus of all points equidistant from B and C. Therefore it

includes the points K and S.

QQ' is the locus of all points equidistant from A and B. Therefore it passes through J and M.

The join of any two points within our square $a^2/4$ must be less than $a\sqrt{2}$.

If we join M and S to B we will form an angle of 60° . (Triangle ABM and triangle BCS are isosceles, having base angles of 15° . Therefore angle MBS = 60° .) As M and S are symmetrically situated, $BM = BS$, and triangle BMS is equilateral.

Then MSR' cannot be equilateral as R' is at a greater distance than B from MS. (Distance of B from MS is $a/\sqrt{2}$ and the distance of R from MS is $a/\sqrt{2}$)

To form an equilateral triangle, R' must be joined to two interior points which are symmetrically situated with respect to BR', the line of symmetry. J,K; P,X; Z,O; L,T are impossible as $MR'S < 60^\circ$, and these points all lie within this angle.

nTherefore type a_1 as a third vertex will not give an equilateral triangle.

Type b_1 as a third vertex will not give an equilateral triangle when combined with two interior points. M', for example, is over b distance to any interior point. We would have a triangle with two sides greater

than b and one side less than $a\sqrt{2}$ (or b).

Type c_3 will not give an equilateral triangle. K_1 , for example, when joined to any point above BK_1 forms an angle with BK_1 at K_1 which is less than 30° . K_2 joined to S forms an angle less than 30° with BK_2 . Therefore the angle at K_2 of the triangle formed by joining K_1 to two interior points is less than 60° .

Therefore as the triangle cannot be equilateral, the type is non-equilateral.

Therefore all types of triangles having two vertices within the original triangle ABC and one vertex of type $a_1, b_2, c_2, a_3, b_3, c_3$, or d_3 are non-equilateral.

Q.E.D.

Theorem 5.

Elimination by consideration of special types.

1. Type a, a_2, a_2 is non-equilateral.

Under type a, a, a we have three possible triangles, $XR'P'$, $RX'P'$, and $PX'R'$. (See page 7 or any figure)

By an independent proof we will show that $P'X'R'$ is equilateral. Then angle $P'X'R' = R'P'X' = P'R'X' = 60$. As X is within triangle $P'R'X'$, angle $P'XR' = 60$, and triangle $P'XR'$ is non-equilateral. In the same way triangles $RX'P'$ and $PX'R'$ may be proved non-equilateral. In like manner we can show that types (x, a_2, a_2) are non-equilateral. $x_1 = a_1, b_1, \text{ or } c_1$.

2. The following types of triangles:- a, a, b_1 ; a, a, c_1 ; a, b, c_1 ; b, b, b_1 ; a, c, c_1 ; b, c, c_1 ; a_2, a_2, b_1 ; a_2, b_2, b_1 ; a_2, a_2, c_2 ; b_2, b_2, b_1 ; a_2, b_2, c_2 ; and b_2, c_2, c_2 , are not to be considered as each triangle of these types has two vertices on the same trisector. See also pages 8, 22. By the complete investigation of interior triangles we find that only the types a, a, a_1 , a, b, b_1 , b, b, c_1 , and c, c, c_1 are possible. By exactly the same method, we can prove that the triangles formed by intersections of exterior trisectors fall into four types. In both cases only the types a, a, a , and a_2, a_2, a_2 are equilateral. Every type may be tested out by one or more methods employed in this paper.

All but Triangle PRX, the only Triangle of
 Type (a, a, a) , of the Interior Triangles
 Are Non-Equilateral.

1 a. Geometrical Diagram Showing the Number of
 Triangles Which May Be Formed by Joining
 Intersections of the Interior Trisectors.

b. Classification of the Eight Interior Triangles.

2 a. Table of Trigonometric Values.

b. Type 1 is Non-Equilateral.

c. Type 2 is Non-Equilateral.

d. Type 3 is Non-Equilateral.

e. Type 4 is Equilateral.

(1) Fallacy of the Geometrical Proof

Offered by A.H.Holmes.

(2) Proof by E.M.Morgan.

Extension of this Work to the Corresponding
 Exterior Triangle P'R'X', the only Triangle of
 Type $a_2 a_2 a_2$.

1. Proof that Triangle P'R'X' is Equilateral.

Geometrical Diagram Showing the Number of
 Triangles Which May Be Formed by Joining
 Intersections of Interior Trisectors.
 Classification of the Eight Triangles.

	1	2	3	4	5	6
1	A	A	M	N	O	P
2	A	A	R	S	T	J
3	M	R	C	C	K	L
4	N	S	C	C	X	Z
5	O	T	K	X	B	B
6	P	J	L	Z	B	B

Any of the twelve points M, N, O, P, R, S, T, J, K, L, X, Z lies on two trisectors. It, therefore, cuts out six other collinear points leaving a choice of five points for the second vertex. Of these five points all will cut out four collinear points which have not been eliminated by M; but one of the five will cut out a fifth point leaving no point for a third vertex, as S. The four remaining points will be paired for the second and third vertices.

Therefore with M as one vertex there are two possible triangles. As there are four points to each

line, the total number of possible triangles is eight. The triangles are MJX, MZT, NJK, ORZ, OLS, PRX, PSK, and LNT.

These eight triangles are of four types as is readily seen when they are written in the notation suggested in the introduction in the classification of the points of intersection.

OZR	(b, b, a,)	Type 1.
SKP	(b, b, a,)	"
MJX	(b, b, a,)	"
JKN	(b, b, c,)	Type 2.
OSL	(b, b, c,)	"
MZT	(b, b, c,)	"
NLT	(c, c, c,)	Type 3.
PRX	(a, a, a,)	Type 4.

In testing out these eight triangles to see which are equilateral, it will be necessary to test only one triangle from each of the four types.

Table of Trigonometric Values Used Constantly
in the Following Proofs.

$$\sin 15^\circ = \cos 75^\circ = \frac{\sqrt{2-\sqrt{3}}}{2}$$

$$\cos 15^\circ = \sin 75^\circ = \frac{\sqrt{2+\sqrt{3}}}{2}$$

$$\sin 30^\circ = \cos 60^\circ = \frac{1}{2}$$

$$\cos 30^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}}$$

$$\sin \frac{1}{2} A = \pm \sqrt{\frac{1-\cos A}{2}}$$

$$\cos \frac{1}{2} A = \pm \sqrt{\frac{1+\cos A}{2}}$$

$$\sin\left(\frac{\pi}{2} \pm A\right) = \pm \cos A$$

$$\cos\left(\frac{\pi}{2} \pm A\right) = \mp \sin A$$

$$\sin(\pi \pm A) = \mp \sin A$$

$$\cos(\pi \pm A) = -\cos A$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\frac{a}{b} = \frac{\sin A}{\sin B}$$

Type 1 is Non-Equilateral.

If we can prove that one triangle of this type is non-equilateral we know that the type is non-equilateral.

We will prove triangle ORZ non-equilateral when triangle ABC is isosceles and right.

If triangle ORZ is equilateral, $OZ = RZ$. See Fig. 1,2,3.

$$OZ^2 = BZ^2 + BO^2 - 2BZ \cdot BO \cos ZBO$$

$$RZ^2 = CZ^2 + RC^2 - 2CZ \cdot RC \cos RCZ$$

$$BZ = BO = a \frac{\sin 15^\circ}{\sin 75^\circ}, \quad CZ = a \frac{\sin 60^\circ}{\sin 75^\circ}, \quad CR = b \frac{\sin 15^\circ}{\sin 30^\circ}$$

$$\therefore \frac{2a^2 \sin^2 15^\circ}{\sin^2 75^\circ} - \frac{2a^2 \sin^2 15^\circ \cos 30^\circ}{\sin^2 75^\circ} =$$

$$\frac{a^2 \sin^2 60^\circ}{\sin^2 75^\circ} - \frac{b^2 \sin^2 15^\circ}{\sin^2 30^\circ} - \frac{2ab \sin 60^\circ \sin 15^\circ \cos 15^\circ}{\sin 75^\circ \sin 30^\circ}$$

$$\frac{8a^2}{2+\sqrt{3}} \frac{2-\sqrt{3}}{4} \left(1 - \frac{3}{2}\right) = \frac{3}{2+\sqrt{3}} + 2(2-\sqrt{3}) - \sqrt{6} \cdot \sqrt{2-\sqrt{3}}$$

$$\text{Reducing, } 4(1+\sqrt{3}) = \sqrt{12-6\sqrt{3}} (2+\sqrt{3}).$$

$$\text{But } 52 \neq -26\sqrt{3}.$$

.. $OZ \neq RZ$ and triangle ORZ is non-equilateral.

.. type 1 is not equilateral.

Q.E.D.

Type 2 is Non-Equilateral.

To test this type we will choose any one of the three triangles OSL, JKN, or MZT, and prove that it is not equilateral.

Let triangle ABC be an isosceles right triangle.

If triangle OSL is equilateral, $OS = SL$.

$$OS^2 = OA^2 + AS^2 - 2 OA \cdot AS \cos OAS$$

$$SL^2 = SC^2 + LC^2 - 2 SC \cdot LC \cos SCL$$

$$OA = a \frac{\sin 60^\circ}{\sin 75^\circ} = a \sqrt{3} \cdot \sqrt{2-\sqrt{3}}$$

$$AS = b \frac{\sin 30^\circ}{\sin 45^\circ} = a$$

$$\cos OAS = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$SC = b \frac{\sin 15^\circ}{\sin 45^\circ} = a \sqrt{2-\sqrt{3}}$$

$$\cos SCL = \cos 15^\circ = \frac{\sqrt{2+\sqrt{3}}}{2}$$

$$LC = a \frac{\sin 60^\circ}{\sin 90^\circ} = a \frac{\sqrt{3}}{2}$$

If $OS^2 = SL^2$,

$$3a^2(2-\sqrt{3}) + a^2 - 2a^2\sqrt{3}\sqrt{2-\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = a^2(2-\sqrt{3}) \quad \frac{3a^2}{4} - \frac{a^2\sqrt{3}\sqrt{2+\sqrt{3}}\sqrt{2-\sqrt{3}}}{2}$$

Reducing, we have $109 = 60\sqrt{3}$.

.. $OS \neq SL$, and triangle OSL is not equilateral.

An interesting case is this; when triangle ABC is isosceles and right, triangle JKN is equilateral.

.. type 2 is non-equilateral.

Q.E.D.

Type 3 is Non-Equilateral.

To test this type we will choose the triangle LNT of this type.

Let triangle ABC be isosceles and right.

If triangle LNT is equilateral, $LT^2 = NT^2$.

$$LT^2 = BL^2 + BT^2 - 2BL \cdot BT \cos \beta$$

$$NT^2 = AN^2 + AT^2 - 2AN \cdot AT \cos \alpha$$

$$BL = \frac{a \sin 30^\circ}{\sin 90^\circ} = \frac{a}{2},$$

$$BT = \frac{a \sin 30^\circ}{\sin 90^\circ} = \frac{a}{2},$$

$$AN = \frac{b \sin 30^\circ}{\sin 60^\circ} = a\sqrt{\frac{2}{3}},$$

$$AT = \frac{c \sin 60^\circ}{\sin 90^\circ} = \frac{a\sqrt{3}}{2}$$

\therefore if $LT^2 = NT^2$,

$$\frac{a^2}{4} + \frac{a^2}{4} - 2 \frac{a^2}{4} \cdot \frac{\sqrt{3}}{2} = a^2 \frac{2}{3} - a^2 \frac{3}{4} - 2a^2 \sqrt{\frac{2}{3}} \sqrt{\frac{3}{2}} \frac{\sqrt{2+\sqrt{3}}}{2},$$

Reducing, $2 = 3\sqrt{3}$, and $LT \neq NT$.

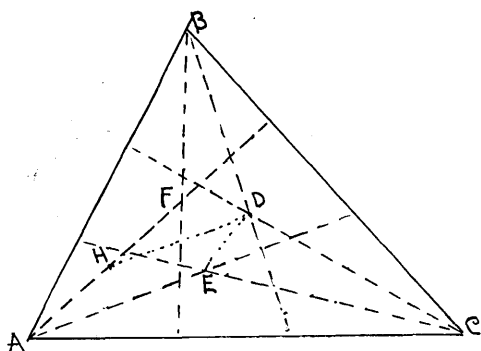
Therefore as triangle LNT is not equilateral in this special case when triangle ABC is isosceles and right, it cannot be equilateral in the general case.

Q.E.D.

The Only Triangle of Type 4 is Equilateral.

The following geometrical solution of the problem proposed by R.C.Archibald of Brown University, was offered by A.H.Holmes and published in the American Mathematical Monthly, Dec. 1910, Vol.17, No.12.

Let ABC be a triangle with AE and AF trisectors of BAC; BF and BD of ACB; and CE and CD of ACB. Draw DE, DF, and EF. Triangle DEF is equilateral.



From D draw DG parallel to AF, cutting AE in G.

Take on AF, AH = DG.

Then $\angle GDH = \angle EAF = \frac{1}{3} \angle BAC$.

Suppose triangle GED be

moved so that GE is collinear

with AF and point E is at F. Then since DG is parallel to AF, and $\angle EAF = \angle FAB$, DG will be parallel to AB.

\therefore triangles DEG and AFB are similar and $\angle EDG = \angle ABF = \frac{1}{3} \angle ABC$. (The proof fails at this point. Triangle DEG has not been proved similar to triangle AFB)

Similarly $\angle FDH = \frac{1}{3} \angle ACB$.

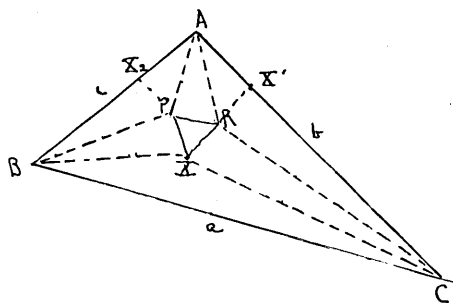
$\therefore \angle EDF = \frac{1}{3} (\angle ABC + \angle ACB + \angle BAC) = 60^\circ$.

In like manner $\angle DEF = 60^\circ$.

\therefore triangle DEF is equilateral.

The Only Triangle of Type 4 is Equilateral.

Triangle PRX is equilateral.



(Original proof by E.M.

Morgan was shortened by

Prof.S.L.Lefschetz)

Given the triangle ABC

with the three interior
angles trisected.

Prove that the triangle is equilateral, whose
vertices are the intersections of trisectors adjacent
to the sides a, b, c.

Construct $CX' = CX$.

Then $RX = RX'$

$$RX'^2 = AR^2 + AX'^2 - 2AX'AR \cos \alpha$$

$$RX^2 = AR^2 + (b - CX)^2 - 2AR(b - CX) \cos \alpha.$$

$$PR^2 = AP^2 + AR^2 - 2AP \cdot AR \cdot \cos \alpha.$$

$$\begin{aligned} RX^2 - PR^2 &= (b - CX)^2 - AP^2 - 2AR \cos \alpha \cdot (b - CX - AP) \\ &= (b - CX - AP)(b - CX + AP - 2AR \cos \alpha). \end{aligned}$$

$$\text{But } CX = \frac{a \sin \beta}{\sin(\beta + \gamma)},$$

$$AP = \frac{c \sin \beta}{\sin(\alpha + \beta)}$$

$$c = \frac{a \sin 3\gamma}{\sin 3\alpha}, \quad b = \frac{a \sin 3\beta}{\sin 3\alpha}, \quad AR = \frac{b \sin \gamma}{\sin(\alpha + \gamma)}$$

$$\text{If } (b - CX + AP - 2AR \cos \alpha) = 0, \quad RX^2 - PR^2 = 0, \text{ or } RX = PR.$$

$$\begin{aligned} b - CX + AP - 2AR \cos \alpha &= \left(b - \frac{a \sin \beta}{\sin(\beta + \gamma)} + \frac{c \sin \beta}{\sin(\alpha + \beta)} - \frac{2 \sin \gamma \cos \alpha}{\sin(\alpha + \gamma)} b \right) \\ &= b \left[\left\{ 1 - \frac{2 \sin \gamma \cos \alpha}{\sin(\alpha + \gamma)} \right\} - \frac{c \sin \beta}{\sin(\alpha + \beta)} - \frac{a \sin \beta}{\sin(\beta + \gamma)} \right] \end{aligned}$$

$$= b \frac{\sin(\alpha+\gamma) - 2\sin\gamma \cos\alpha}{\sin(\alpha+\gamma)} + \frac{c \sin\beta}{\sin(\alpha+\beta)} - \frac{a \sin\beta}{\sin(\beta+\gamma)}$$

$$= \frac{b \sin(\alpha-\gamma)}{\sin(\alpha+\gamma)} + \frac{c \sin\beta}{\sin(\alpha+\beta)} - \frac{a \sin\beta}{\sin(\beta+\gamma)}$$

$$\text{as } \sin(\alpha+\gamma) - 2 \sin\gamma \cdot \cos\alpha = \sin(\alpha-\gamma).$$

$$= \frac{a \sin 3\beta \sin(\alpha-\gamma)}{\sin 3\alpha \sin(\alpha+\gamma)} + \frac{a \sin 3\gamma \sin\beta}{\sin 3\alpha \sin(\alpha+\beta)} - \frac{a \sin\beta}{\sin(\beta+\gamma)}$$

$$= \frac{a}{\sin 3\alpha} \left[\frac{\sin 3\beta \sin(\alpha-\gamma)}{\sin(\alpha+\gamma)} + \frac{\sin 3\gamma \sin\beta}{\sin(\alpha+\beta)} - \frac{\sin 3\alpha \sin\beta}{\sin(\beta+\gamma)} \right]$$

$$\text{As } \sin 3\theta = 3 \sin\theta - 4 \sin^3\theta,$$

$$\sin 3\beta = \sin 3(\alpha+\gamma) = 3 \sin(\alpha+\gamma) - 4 \sin^3(\alpha+\gamma).$$

$$= \frac{a}{\sin 3\alpha} \left[\sin(\alpha-\gamma) \{3-4\sin^2(\alpha+\gamma)\} - \sin\beta \{3-4\sin^2(\alpha+\beta) - 3+4\sin^2(\beta+\gamma)\} \right]$$

$$= \frac{a}{\sin 3\alpha} \left[\sin(\alpha-\gamma) \{3-4\sin^2(\alpha+\gamma)\} - 4\sin\beta \left\{ \frac{1-\cos 2(\alpha+\beta)}{2} - \frac{1-\cos 2(\beta+\gamma)}{2} \right\} \right]$$

$$= \frac{a}{\sin 3\alpha} \left[\sin(\alpha-\gamma) \{3-4\sin^2(\alpha+\gamma) - 2\sin\beta \{ \cos 2(\beta+\gamma) - \cos 2(\alpha+\beta) \} \} \right]$$

$$\text{As } \cos(A-B) - \cos(A+B) = 2 \sin A \cdot \sin B$$

$$\text{If } A = \alpha + 2\beta + \gamma.$$

$$A-B = 2\alpha + 2\beta.$$

$$B = \alpha - \gamma$$

$$A-B = 2\beta + 2\gamma.$$

$$\cos(A-B) - \cos(A+B) = 2 \sin A \cdot \sin B.$$

$$= \frac{a}{\sin 3\alpha} \left[\sin(\alpha-\gamma) \{3-4 \sin^2(\alpha+\gamma)\} - 2 \sin\beta \{ \cos 2(\beta+\gamma) - \cos 2(\alpha+\beta) \} \right].$$

$$= \frac{a}{\sin 3\alpha} \left[\sin(\alpha-\gamma) \{3-4\sin^2(\alpha+\gamma)\} - 4\sin\beta \{ \sin(\alpha-\gamma) \sin(\alpha+2\beta+\gamma) \} \right]$$

$$= \frac{a \sin(\alpha-\gamma)}{\sin 3\alpha} \left[3-4\sin^2(\alpha+\gamma) - 4\sin\beta \sin(\alpha+2\beta+\gamma) \right].$$

$$= \frac{a \sin(\alpha-\gamma)}{\sin 3\alpha} \left[1 + 2 \left\{ 1 - 2\sin^2(\alpha+\gamma) - 2\sin\beta \cdot \sin(\alpha+2\beta+\gamma) \right\} \right]$$

But $1 - 2\sin^2(\alpha+\gamma) = \cos 2(\alpha+\gamma)$ and

$$-2\sin\beta \cdot \sin[2\beta+(\alpha+\gamma)] = 2 \left[\cos(3\beta+\alpha+\gamma) - \cos\beta \cdot \cos(\alpha+2\beta+\gamma) \right]$$

$$= \frac{a \sin(\alpha-\gamma)}{\sin 3\alpha} \left[1 + 2 \left\{ \cos 2(\alpha+\gamma) + 2\cos(\alpha+3\beta+\gamma) - 2\cos\beta \cos(\alpha+2\beta+\gamma) \right\} \right]$$

As $\cos 2(\alpha+\gamma) + 2\cos(\alpha+3\beta+\gamma) - 2\cos\beta \cdot \cos(\alpha+2\beta+\gamma)$

$$= \cos(120^\circ - 2\beta) + 2\cos(60^\circ + 2\beta) - 2\cos\beta \cdot \cos(60^\circ + \beta)$$

for $\alpha+\beta+\gamma = 60^\circ$.

But $2\cos\beta \cdot \cos(60^\circ + \beta) = \cos(2\beta + 60^\circ) + \cos 60^\circ$, as

$$2\cos\beta \cdot \cos(60^\circ + \beta) = 2\cos\beta \left[\cos\beta \cos 60^\circ - \sin\beta \sin 60^\circ \right] =$$

$$= (\cos 2\beta + 1)\cos 60^\circ - \sin 2\beta \cdot \sin 60^\circ = \cos(2\beta + 60^\circ) + \cos 60^\circ.$$

$$\begin{aligned} \therefore \cos(120^\circ - 2\beta) + 2\cos(60^\circ + 2\beta) - 2\cos\beta \cdot \cos(60^\circ + \beta) &= \\ &= \cos(120^\circ - 2\beta) + 2\cos(60^\circ + 2\beta) - \cos(60^\circ + 2\beta) - \cos 60^\circ \\ &= \cos(120^\circ - 2\beta) - \cos[180^\circ - (60^\circ + 2\beta)] - \cos 60^\circ \\ &= \cos(120^\circ - 2\beta) - \cos(120^\circ - 2\beta) - 1/2 = -1/2. \end{aligned}$$

$$\therefore \frac{a \sin(\alpha-\gamma)}{\sin 3\alpha} \left[1 + 2(-1/2) \right] = 0.$$

\therefore factor $(b - CX + AP - 2AR \cos \alpha)$, and $RX^2 - PR^2 = 0$, or

$$RX = PR.$$

In like manner $PX = RX = PR$.

\therefore triangle PRX is equilateral.

Q.E.D.

All but Triangle P'R'X', the Only Triangle
of Type (a_1, a_2, a_3) of the Intersections of
Exterior Trisectors Are Non-Equilateral.

The geometrical diagram showing the number of
triangles which may be formed by joining intersections
of interior trisectors will, by adding primes to the
notation, exactly fit the case of the exterior trisectors.

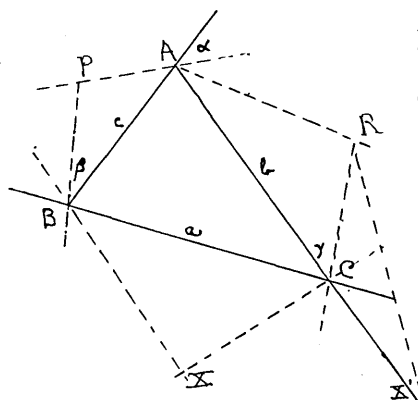
The eight triangles which may be formed by joining
intersections of exterior trisectors, will follow the
same classification into the four types.

By exactly the same method the first three types
may be proved non-equilateral.

The proof that triangle P'R'X', the only triangle
of type 4, (a_1, a_2, a_3) , is similar to that given for
triangle PRX.

Type 4, (a_1, a_2, a_3) is Equilateral.

PRX is an equilateral triangle.



Extend AC and let $CX_1 = CX$

Triangles CRX and CRX_1 are

equal, as $RCX = RCX_1$,

(Two sides and included angle of one are equal to two sides and included angle of the other)

$$RX^2 = AR^2 + (b + CX_1)^2 - 2 \cdot AR(b + CX_1) \cos \alpha.$$

$$RX = RX_1$$

$$PR^2 = AR^2 + AP^2 - 2 AR AP \cos(A + 2\alpha).$$

$$\cos(A + 2\alpha) = \cos(\pi - \alpha) = -\cos \alpha.$$

$$RX^2 - PR^2 = (b + CX)^2 - AP^2 - 2AR \cdot \cos \alpha \cdot (b + CX + AP)$$

$$= (b + CX + AP)(b + CX - AP - 2AR \cdot \cos \alpha)$$

$$CX = \frac{a \sin \beta}{\sin(\beta + \gamma)}, \quad AP = \frac{c \sin \beta}{\sin(\alpha + \beta)}, \quad AR = \frac{b \sin \gamma}{\sin(\alpha + \gamma)},$$

$$b = \frac{a \sin B}{\sin A} = \frac{a \sin 3\beta}{\sin 3\alpha}, \quad c = \frac{a \sin 3\gamma}{\sin 3\alpha}$$

$$\text{If } (b + CX - AP - 2 AR \cos \alpha) = 0, \quad RX = PR.$$

$$b + CX - AP - 2 \cdot AR \cdot \cos \alpha = b - \frac{2b \cdot \cos \alpha \cdot \sin \gamma}{\sin(\alpha + \gamma)} + \frac{a \sin \beta}{\sin(\beta + \gamma)} - \frac{c \sin \beta}{\sin(\alpha + \beta)}$$

$$= b \left(1 - \frac{2 \sin \gamma \cos \alpha}{\sin(\alpha + \gamma)} \right) - \frac{c \sin \beta}{\sin(\alpha + \beta)} + \frac{a \sin \beta}{\sin(\beta + \gamma)}$$

$$= \frac{b \sin(\alpha - \gamma)}{\sin(\alpha + \gamma)} - \frac{c \sin \beta}{\sin(\alpha + \beta)} + \frac{a \sin \beta}{\sin(\beta + \gamma)}$$

$$\begin{aligned}
&= \frac{a \sin 3\beta \sin(\alpha-\gamma)}{\sin 3\alpha \sin(\alpha+\gamma)} - \frac{a \sin 3\gamma \sin\beta}{\sin 3\alpha \sin(\beta+\gamma)} + \frac{a \sin\beta}{\sin(\beta+\gamma)} \\
&= \frac{a}{\sin 3\alpha} \left[\sin(\alpha-\gamma) \{3-4\sin^2(\alpha+\gamma)\} - \sin\beta \{3-4\sin^2(\beta+\gamma)\} - 3 + 4 \sin(\alpha+\beta) \right] \\
&= \frac{a}{\sin 3\alpha} \left[\sin(\alpha-\gamma) \{3-4\sin^2(\alpha+\gamma)\} + 4\sin\beta \{ \sin^2(\beta+\gamma) - \sin(\alpha+\beta) \} \right] \\
&= \frac{a}{\sin 3\alpha} \left[\sin(\alpha-\gamma) \{ \{3-4\sin^2(\alpha+\gamma)\} + 4 \sin\beta \cdot \sin(\alpha+2\beta+\gamma) \} \right] \\
&= \frac{a \sin(\alpha-\gamma)}{\sin 3\alpha} \left[1 + 2 \{ 1 - 2\sin^2(\alpha+\gamma) + 2 \sin\beta \sin(\alpha+2\beta+\gamma) \} \right]
\end{aligned}$$

But $1-2\sin^2(\alpha+\gamma) = \cos 2(\alpha+\gamma)$, and

$$2\sin\beta \cdot \sin(2\beta+\alpha+\gamma) = -2 [\cos(3\beta+\alpha+\gamma) - \cos\beta \cdot \cos(\alpha+2\beta+\gamma)]$$

$$= \frac{a \sin(\alpha-\gamma)}{\sin 3\alpha} \left[1 + 2 \{ \cos 2(\alpha+\gamma) - 2\cos(\alpha+3\beta+\gamma) + 2\cos\beta \cos(\alpha+2\beta+\gamma) \} \right]$$

$$\begin{aligned}
&= \frac{a \sin(\alpha-\gamma)}{\sin 3\alpha} \left[1 + 2 \cdot \cos 2(\alpha+\gamma) - 2 \cos(\alpha+3\beta+\gamma) - \right. \\
&\quad \left. - 2 \cos(\alpha+3\beta+\gamma) + 4 \cos\beta \cdot \cos(\alpha+2\beta+\gamma) \right]
\end{aligned}$$

$$\text{As } -\cos(\alpha+3\beta+\gamma) + 2\cos\beta \cdot \cos(\alpha+2\beta+\gamma) = \cos(\alpha+\beta+\gamma)$$

$$= \frac{a \sin(\alpha-\gamma)}{\sin 3\alpha} \left[1 + 2\cos 2(\alpha+\gamma) - 2\cos(\alpha+3\beta+\gamma) + 2 \cos(\alpha+\beta+\gamma) \right]$$

$$\text{But } 2\pi - 3\alpha - 3\gamma = 3\beta, \quad \cos\theta = \cos(2\pi-\theta),$$

$$\cos(\alpha+3\beta+\gamma) = \cos 2(\alpha+\gamma).$$

$$= \frac{a \sin(\alpha-\gamma)}{\sin 3\alpha} \left[1 + 2\cos 2(\alpha+\gamma) - 2\cos 2(\alpha+\gamma) + 2\cos(\alpha+\beta+\gamma) \right]$$

$$\text{As } \cos(\alpha+\beta+\gamma) = \cos 120^\circ = -1/2.$$

$$= \frac{a \sin(\alpha-\gamma)}{\sin 3\alpha} \left[1 + 2 \cdot \cos 120^\circ = 0 \quad \therefore RX = PR. \right]$$

In like manner we can prove that $RX = PX = PR$.

Q.E.D.

Table Calculating the Number of Triangles
Remaining.

3(a₁) or (a, a, a₁)

2(a₁) 1(b₁) or (a, a, b₁) Eliminated by Th.5.
 1(c₁) (a, a, c₁) "

Theorem 4 eliminates (a, a, a₂), etc.

1(a₁) 2(b₁) Eliminated by Th.5.
 2(c₁) "
 2(a₂) "
 2(b₂)
 2(a₃)
 2(c₃)
 2(d₃)

1(b₁) 1(c₁) or (a, b, c₁) Eliminated by Th.5, 2.
Theorem 4 eliminates exterior points.

1(a₂) 1(b₂)
 1(a₃)
 1(c₃)
 1(d₃)

1(b₂) 1(a₃)
 1(c₃)
 1(d₃)

1(a₃) 1(c₃)
 1(d₃)

1(c₃) 1(d₃)

3(b₁) Eliminated by Th5.

2(b₁) 1(c₁) "

1(b₁) 2(c₁) "
 2(a₂) "
 2(b₂)
 2(a₃)
 2(c₃)
 2(d₃)

$1(b_1)$	$1(a_1)$	$1(b_1)$ $1(a_3)$ $1(c_3)$ $1(d_3)$
----------	----------	--

	$1(b_2)$	$1(a_3)$ $1(c_3)$ $1(d_3)$
--	----------	----------------------------------

	$1(a_3)$	$1(c_3)$ $1(d_3)$
--	----------	----------------------

	$1(c_3)$	$1(d_3)$
--	----------	----------

$3(c_1)$

$2(c_1)$ Exterior points are eliminated by Th.4.

$1(c_1)$	$2(a_2)$ $2(b_2)$ $2(a_3)$ $2(c_3)$ $2(d_3)$	Eliminated by Th.5.
----------	--	---------------------

$1(c_1)$	$1(a_1)$	$1(b_2)$ $1(a_3)$ $1(c_3)$ $1(d_3)$
----------	----------	--

	$1(b_2)$	$1(a_3)$ $1(c_3)$ $1(d_3)$
--	----------	----------------------------------

	$1(a_3)$	$1(c_3)$ $1(d_3)$
--	----------	----------------------

	$1(c_3)$	$1(d_3)$
--	----------	----------

$3(a_1)$

$2(a_1)$	$1(b_2)$ $1(a_3)$ $1(c_3)$ $1(d_3)$	Eliminated by Th.5.
----------	--	---------------------

$1(a_2)$ $2(b_2)$ Eliminated by Th.5.

$2(a_3)$
 $2(c_3)$
 $2(d_3)$

$1(a_2)$ $1(b_1)$ $1(a_3)$
 $1(c_3)$
 $1(d_3)$

$1(a_3)$ $1(c_3)$
 $1(d_3)$

$1(c_3)$ $1(d_3)$

$3(b_1)$ Eliminated by Th.5.

$2(b_2)$ $1(a_3)$
 $1(c_3)$
 $1(d_3)$

$1(b_1)$ $2(a_3)$
 $2(c_3)$
 $2(d_3)$

$1(a_3)$ $1(c_3)$
 $1(d_3)$

$1(c_3)$ $1(d_3)$

$3(a_3)$

$2(a_3)$ $1(c_3)$
 $1(d_3)$

$1(a_3)$ $2(c_3)$
 $2(d_3)$
 $1(c_3)$ $1(d_3)$

$3(c_3)$
 $2(c_3)$ $1(d_3)$
 $1(c_3)$ $2(d_3)$

$3(d_3)$

∴ the total number of types of triangles not eliminated is 76.

Conclusion.

Although we have seventy six types which have not been eliminated, without doubt, many of them can readily be proved non-equilateral by the various methods employed.

It is our belief that only two triangles of this number can be proved equilateral.

We have proved that only one of the interior triangles can be equilateral.

We have also proved the corresponding exterior triangle equilateral.